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Characterizations of Native Spaces

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Abstract. In the theory of radial basis functions, linear combinations of the translates of a single function Φ are used as interpolants. The space spanned by all of these linear combinations carries an inner product defined via Φ itself. It can be completed and becomes a Hilbert space, called the native space for Φ, which is of great importance for further investigation of radial basis functions. The native space will contain abstract elements which are not linear combinations of radial basis functions, and require some work to be recognized as functions. This paper provides some characterizations of native spaces and relates some of the different approaches used to define them. Finally, embedding results for native spaces into Sobolev spaces are proven.

§1. Introduction

Our goal is to describe properties of the set of functions

$$\sum_{j=1}^{N} c_j \Phi(x, x_j), \quad x \in \Omega, \quad c_j \in \mathbb{C}, \tag{1}$$

where Ω is a subset of $\mathbb{R}^d$ and Φ is a real-valued symmetric function on $\Omega \times \Omega$. These functions depend on sets $X = \{x_1, \ldots, x_N\} \subset \Omega$ of N pairwise distinct points called "centers", while the number N of centers and their placement within Ω are arbitrary. Functions of the form (1) arise naturally as tools for multivariate approximation, especially if Φ is a radial basis function $\Phi(x, y) := \phi(\|x - y\|_2)$ with a real-valued function ϕ on $[0, \infty)$. We shall study the closure of the linear span of functions (1) under a natural topology that comes from Φ itself, provided that Φ has a crucial property:

Definition 1. *A function $\Phi \in C(\Omega \times \Omega)$ is called* conditionally positive definite *(abbreviated as* **c.p.d.***) of order m on Ω if the quadratic form*

$$\sum_{j,k=1}^{N} c_j \bar{c}_k \Phi(x_j, x_k)$$

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is positive for all sets $X = \{x_1, \dots, x_N\} \subset \Omega$ *of* N *pairwise distinct points and all vectors* $c = (c_1, \dots, c_N)^T \in \mathbb{C}^N \setminus \{0\}$ *satisfying*

$$\sum_{i=1}^{N} c_i p(x_i) = 0 \text{ for all } p \in P_m^d, \tag{2}$$

where P_m^d *is the space of d-variate complex-valued polynomials of order not exceeding* m.

There are various possibilities to proceed from here. Already in their early pioneering papers, Madych and Nelson already took two different approaches, via finitely supported functionals [3] and via a specific version of generalized Fourier transforms [4] in the spirit of Gelfand–Shilov. The latter requires measure–theoretic arguments at certain places, and is rather complicated to deal with. The dissertation of Iske [1] used variational inequalities, while Weinrich [6] proceeded via regularized distributions in the sense of Schwartz. Our goal here is to show, as far as possible, the equivalence of the cited approaches. Since the access via generalized Fourier transforms has problems in dealing with arbitrary domains $\Omega \subseteq \mathbb{R}^d$, we proceed as in [5] in order to start with the most general approach known so far.

§2. Construction via Finitely Supported Functionals

Consider the space

$$(P_m^d)_\Omega^\perp := \left\{ \sum_{i=1}^{N} c_i \delta_{x_i} \middle| \; c_i \in \mathbb{C},\; x_i \in \Omega \text{ for } 1 \le i \le N \text{ with } (2) \right\}$$

of all functionals that are finitely supported in Ω and vanish on the polynomials in P_m^d. Starting with a c.p.d. function Φ of order m in Ω, we define

$$(\lambda, \mu)_\Phi := \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_i \bar{\mu}_j \Phi(x_i, y_j)$$

for $\lambda, \mu \in (P_m^d)_\Omega^\perp$ with $\lambda = \sum_{i=1}^{N} \lambda_i \delta_{x_i}$, $\mu = \sum_{j=1}^{M} \mu_j \delta_{y_j}$ to get an inner product $(\cdot,\cdot)_\Phi$ which induces a Φ–dependent norm in the Φ–independent space $(P_m^d)_\Omega^\perp$. To relate functionals with functions, we use the map

$$R_\Phi \; : \; (P_m^d)_\Omega^\perp \to C(\Omega), \; R_\Phi(\lambda) := \lambda^x \Phi(x, \cdot) =: \lambda * \Phi,$$

where λ^x stands for the action of λ with respect to the variable x. By standard Hilbert space arguments, the fundamental identity

$$\mu(R_\Phi(\lambda)) = (\lambda, \mu)_\Phi \text{ for all } \lambda, \mu \in (P_m^d)_\Omega^\perp \tag{3}$$

proves that $(P_m^d)_\Omega^\perp$ and its image under R_Φ form a dual pair. Furthermore, this equation carries over to the Hilbert space closures $P_{\Phi,\Omega}$ of $(P_m^d)_\Omega^\perp$ and $F_{\Phi,\Omega}$ of $R_\Phi((P_m^d)_\Omega^\perp)$, respectively.

This construction is simple, but it leads to rather abstract elements instead of classical functionals and functions. To overcome this problem, one assumes that P_m^d and Ω allow a Lagrange–type basis $l_1, \ldots, l_{m'}$ with $m' = \dim P_m^d$ and points $x_1, \ldots, x_{m'} \in \Omega$ such that $l_i(x_j) = \delta_{ij}$ for $1 \le i, j \le m'$. Then the functional $\delta_{(x)} := \delta_x - \sum_{i=1}^{m'} l_i(x)\delta_{x_i}$ lies in $(P_m^d)_\Omega^\perp$, and the map S_Φ with

$$S_\Phi(\mu)(x) := (\mu, \delta_{(x)})_\Phi = \delta_{(x)} R_\Phi(\mu) = \mu(R_\Phi(\delta_{(x)})) \text{ for all } \mu \in P_{\Phi,\Omega}, x \in \Omega$$

uses (3) to define a classical function $S_\Phi(\mu)$ for each abstract element $\mu \in P_{\Phi,\Omega}$. The space $G_{\Phi,\Omega} := S_\Phi((P_m^d)_\Omega^\perp)$ now is a much more concrete space. The first of our results can be found in [2] with full proofs.

Theorem 2. *The spaces $F_{\Phi,\Omega} := R_\Phi(P_{\Phi,\Omega})$ and $G_{\Phi,\Omega} := S_\Phi(P_{\Phi,\Omega})$ are isometrically isomorphic via the mapping $S_\Phi \circ R_\Phi^{-1}$ and the inner product it introduces on $G_{\Phi,\Omega}$. Furthermore,*

$$\mu(S_\Phi(\lambda)) = (\lambda, \mu)_\Phi = \mu(R_\Phi(\lambda)) \tag{4}$$

holds for all functionals in $(P_m^d)_\Omega^\perp$ and its closure $P_{\Phi,\Omega}$.

It is not straightforwardly possible to associate classical function values to the elements $R_\Phi(\lambda)$. But (4) indicates that $R_\Phi(\lambda)$ and $S_\Phi(\lambda)$ should agree up to a polynomial from P_m^d on Ω. The function $S_\Phi(\lambda)$, however, vanishes on the points we used for the Lagrange interpolation in P_m^d, and thus realizes a very special assignment of function values modulo P_m^d. Thus, we can interpret $R_\Phi(\lambda)$ as an equivalence class of functions mod P_m^d on Ω, one representer of which is $S_\Phi(\lambda)$. Thus we should add P_m^d to the spaces we dealt with so far.

Definition 3. *Let Φ be c.p.d. of order $m \ge 0$ in Ω. Then the direct sum*

$$N_\Phi(\Omega) := P_m^d(\Omega) \oplus G_{\Phi,\Omega}$$

is called the native space *of Φ.*

The above construction allows us to define a semi–inner product $(.,.)_\Phi$ on this space such that the nullspace is P_m^d. Theorem 2 now implies the isometric isomorphisms $N_\Phi(\Omega) \cong P_m^d(\Omega) \oplus P_{\Phi,\Omega}$ and $N_\Phi(\Omega) \cong P_m^d(\Omega) \oplus F_{\Phi,\Omega}$ as two characterizations of the native space. We add two others, with proofs in [2] dating partially back to [3]:

Theorem 4. *Assume Ω is a subset of $\mathbb{R}^d$ and $m \ge 0$. Then $N_\Phi(\Omega)$ is the unique subspace of $C(\Omega)$ with a semi-inner product $(\cdot,\cdot)_\Phi$ satisfying*

(a) the null-space of the semi-norm is $P_m^d(\Omega)$,

(b) $N_\Phi(\Omega)/P_m^d(\Omega)$ is a Hilbert space,

(c) if $\mu \in (P_m^d)_\Omega^\perp$, then $\mu * \Phi \in N_\Phi(\Omega)$ and $(\mu * \Phi, f)_\Phi = \mu(\bar{f})$ for all $f \in N_\Phi(\Omega)$.

Theorem 5. *Fix $m \geq 0$ and a c.p.d. function Φ of order m in Ω. Then a complex-valued function f on Ω is in $N_\Phi(\Omega)$ iff there is a constant $c(f)$ such that*

$$|\mu(\bar{f})| \leq c(f)\|\mu\|_\Phi$$

for all μ in $(P_m^d)_\Omega^\perp$. The smallest possible constant for such f is the seminorm $\|f\|_\Phi$.

The following sections will proceed gradually from here to other characterizations of native spaces. The main guideline is the various forms that functionals can take, starting from finitely supported functionals used in this section. We proceed via measures (finitely or compactly supported) to distributions, and we refer the reader to [2] for full proofs.

§3. Construction via Measures

Definition 6. *The family of all finitely supported measures on Ω is denoted by $M(\Omega)$.*

Theorem 7. *Let m be a nonnegative integer. Assume Φ is positive definite in Ω with the following property: for all $\lambda \in M(\Omega)$ and $\varepsilon > 0$, there exists μ_ε in $(P_m^d)_\Omega^\perp$ satisfying*

$$\|\mu_\varepsilon - \lambda\|_\Phi < \varepsilon.$$

Then $(P_m^d)_\Omega^\perp$ is contained in $M(\Omega)$, and $M(\Omega)$ is isometrically isomorphic to a dense subset of $\overline{(P_m^d)_\Omega^\perp}$. Furthermore, we have

$$N_\Phi(\Omega) \cong P_m^d(\Omega) \oplus \overline{M(\Omega)},$$

where the closure is induced by Φ. The inner product on $\overline{M(\Omega)}$ is defined as $(\lambda, \mu)_\Phi := \lambda(\overline{\mu * \overline{\Phi}})$.

Now we introduce a new space $\langle(P_m^d)_\Omega^\perp\rangle$ consisting of all compactly supported measures μ on Ω with vanishing moments for P_m^d, i.e., all integrals of polynomials from P_m^d with respect to μ are zero. If we assume

$$\int_\Omega \int_\Omega \Phi(x,y) d\nu(y) d\overline{\mu(x)} = \int_\Omega \int_\Omega \Phi(x,y) d\overline{\mu(x)} d\nu(y)$$

for all μ, ν in $\langle(P_m^d)_\Omega^\perp\rangle$, and

$$\nu(\overline{\nu * \overline{\Phi}}) > 0$$

for all nonzero ν, it is easily checked that

$$(\nu, \mu) := \nu(\overline{\mu * \overline{\Phi}})$$

forms an inner product on $\langle(P_m^d)_\Omega^\perp\rangle$. Then we have the following theorem:

Theorem 8. *Under the above assumptions, $\overline{\langle (P_m^d)_\Omega^\perp \rangle}$ is isometrically isomorphic to $\overline{(P_m^d)_\Omega^\perp}$. Furthermore, the native space $N_\Phi(\Omega)$ is equivalent to $P_m^d(\Omega) \oplus \overline{\langle (P_m^d)_\Omega^\perp \rangle}$.*

The proof of Theorem 8 in [2] is quite hard. It involves *weak** topology and the Krein-Milman theorem. So far, Theorem 8 is the best result concerning interpretation of the dual of the native space as a space of measures.

§4. Construction via Tempered Test Functions

Starting from [4] there is an approach to native spaces via generalized Fourier transforms in the sense of Gelfand and Shilov. Here, we want to avoid distributions and generalized Fourier transforms as far as possible. The key point is to use variational equations on spaces of tempered test functions as a convenient substitute for generalized Fourier transforms.

Let $\mathcal{S}(\Omega)$ denote the space of tempered test functions in the sense of Laurent Schwartz with supports contained in Ω, and define $\mathcal{S}_m^\perp(\Omega)$ as the space of tempered test functions with support in Ω and vanishing moments up to order m. For all $v, w \in \mathcal{S}_m^\perp(\Omega)$

$$\langle v, w \rangle_\Phi := \int_\Omega \int_\Omega \Phi(x,y) v(x) \overline{w(y)} dx dy$$

is a bilinear form, and we would like to base a second construction of the native space on it. To this end, it would be a reasonable possibility to define a property like "tempered conditional positive definiteness" to require that this form is positive definite on $\mathcal{S}_m^\perp(\Omega)$. The result would be a different theory, but we want to blend this approach into our previous setting. Thus we look at conditions that allow to relate this bilinear form to the earlier one.

Following [1], we assume a continuous positive function $\varphi : \mathbb{R}^d \backslash \{0\} \longrightarrow \mathbb{R}$ exists such that

$$\langle v, w \rangle_\Phi = (2\pi)^{-d} \int_{R^d} \varphi(x) \hat{v}(x) \overline{\hat{w}(x)} dx \tag{5}$$

for all $v, w \in \mathcal{S}_m^\perp(\Omega)$. Here

$$\hat{v}(w) := \int_\Omega e^{-ix^T w} v(x) dx$$

denotes the classical Fourier transform of v. By approximation of functionals from $(P_m^d)_\Omega^\perp$ by regular distributions generated by functions from $\mathcal{S}_m^\perp(\Omega)$, Iske [1] proved that this assumption is slightly stronger than c.p.d. of Φ on $\mathbb{R}^d$, and that $(v, w)_\Phi = \langle v, w \rangle_\Phi$ holds for all $v, w \in \mathcal{S}_m^\perp(\Omega)$.

Definition 9. *If a function Φ of difference form $\Phi(x,y) = \phi(x-y)$ with a continuous and even function ϕ on $\mathbb{R}^d$ satisfies (5), we call Φ* variationally positive definite (v.p.d) *of order $m \geq 0$ on $\mathbb{R}^d$.*

Definition 10. *Let Φ be v.p.d. of order $m \geq 0$ on $\mathbb{R}^d$. A complex-valued function f is in the space $\mathcal{C}_{\Phi,m}(\Omega)$ if and only if $f \in C(\Omega)$ and there exists a constant $c(f)$ such that*

$$|\int_\Omega f(x)v(x)dx| \leq c(f)\{\int_\Omega \int_\Omega \Phi(x,y)v(x)\overline{v(y)}dxdy\}^{1/2} \text{ for all } v \in \mathcal{S}_m^\perp(\Omega).$$

Theorem 11. *Let Ω be open and Φ be v.p.d. of order $m \geq 0$ in $\mathbb{R}^d$. Then $N_\Phi(\Omega) \cong \mathcal{C}_{\Phi,m}(\Omega)$. Furthermore, $\overline{\mathcal{S}_m^\perp(\Omega)}$ is isometrically isomorphic to $\overline{(P_m^d)_\Omega^\perp}$.*

Theorem 11 provides a nice unification of the theories of Weinrich [6] and Iske [1]. Their work is based on $(P_m^d)_\Omega^\perp$ and $\mathcal{S}_m^\perp(\Omega)$, respectively. The proof of Theorem 11 is rather involved [2].

§5. Embedding Theorems

We now construct continuous embeddings of native spaces into well-known spaces. Madych and Nelson's discovery that $N_\Phi(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ can be regarded as the first step towards embedding theorems, but it was just an inclusion result. In this paper, all the embedding theorems concern continuous embeddings with respect to the topologies of the spaces. Even the embeddings of native spaces into L_2 spaces can be nontrivial, provided that the underlying domains are unbounded (see [5] for the bounded case).

In this section we first assume Φ to be v.p.d. of order 0 on $\mathbb{R}^d$ with a positive classical Fourier transform $\varphi \in L_1(\mathbb{R}^d)$ of ϕ with $\Phi(x,y) = \phi(x-y)$. All functions f of the form (1) have a classical Fourier transform

$$\widehat{f}(\omega) = \varphi(\omega)\sum_{j=1}^N c_j e^{-i\omega\cdot x_j},$$

and there is an isometry $B : R_\Phi((P_0^d)_\Omega^\perp) \to L_2(\mathbb{R}^d)$, $f \mapsto \widehat{f}/\sqrt{\varphi}$ mapping these functions into $L_2(\mathbb{R}^d)$. It is now easy to see that the equation $\widehat{f} = \sqrt{\varphi}\cdot B(f)$ holds for all functions in $R_\Phi((P_0^d)_\Omega^\perp)$ and its closure $F_{\Phi,\Omega}$ which can be identified with the native space of Φ.

Theorem 12. *For variationally positive definite functions on $\mathbb{R}^d$ of order zero with a positive L_1 Fourier transform φ, the functions in the native space of Φ have Fourier transforms of the form $\sqrt{\varphi}\cdot g$ with an L_2 function g. The native space for Φ can be continuously embedded [2] into $L_2(\mathbb{R}^d)$.*

The last statement was generalized in [2] to

Theorem 13. *Let Φ be symmetric and translation–invariant on $\Omega \times \Omega$ and c.p.d. of order $m \geq 0$ on a domain $\Omega \subseteq \mathbb{R}^d$ containing points $\xi_1, \ldots, \xi_N$ which uniquely determine polynomials of $P_m^d(\Omega)$. If there exists a positive continuous $g \in L^1(\Omega)$ which decays exponentially at infinity and satisfies*

$$\int_\Omega |p(x)\phi(x)|g(x)dx < \infty$$

for all $p(x) \in P_m^d(\Omega)$, then $F_{\Phi,\Omega}$ can be continuously embedded in $L^2(\Omega)$.

Theorem 12 characterizes native spaces as spaces of functions whose Fourier transforms lie in a weighted L_2 space. The same holds for Sobolev spaces on $\mathbb{R}^d$, and this similarity can be used to derive theorems for embedding of native spaces into global Sobolev spaces on $\mathbb{R}^d$. For embeddings of local native spaces on domains $\Omega \subseteq \mathbb{R}^d$, we refer the reader to the fact (proven in [2] and [5]) that functions in native spaces always have an extension to the largest domain where Φ has the c.p.d. property. This yields embeddings of local native spaces into spaces of restrictions of global Sobolev spaces for globally defined functions Φ, but the case of purely locally defined Φ is unsolved.

If Φ is v.p.d.of positive order m on $\mathbb{R}^d$, the function φ of (5) will have a singularity at zero, and thus the notion of Fourier transforms needs generalization. We simply view (5) as a variational property satisfied by the generalized Fourier transform φ of Φ, and we want to prove

Theorem 14. *For v.p.d. functions Φ on $\mathbb{R}^d$ of order $m > 0$ the functions f in the native space of Φ have generalized Fourier transforms $\widehat{f} = \sqrt{\varphi} \cdot g$ with an L_2 function g, where the generalized Fourier transform of f is defined via the variational property*

$$\int f \cdot w = (2\pi)^{-d} \int \widehat{f} \cdot \widehat{w} \text{ for all } w \in \mathcal{S}_m^\perp(\Omega).$$

Proof: We take two functions $v, w \in \mathcal{S}_m^\perp(\Omega)$ and form the function $f_v := \Phi * \bar{v}$. Then (5) yields

$$\begin{aligned}
\int_\Omega w \cdot f_v &= (2\pi)^{-d} \int \varphi \widehat{w} \overline{\widehat{\bar{v}}} \\
&= (2\pi)^{-d} \int \widehat{w} \sqrt{\varphi} \sqrt{\varphi} \overline{\widehat{\bar{v}}} \qquad (6) \\
&= (2\pi)^{-d} \int \widehat{w} \sqrt{\varphi} B(f_v),
\end{aligned}$$

if we define $B(f_v) := \sqrt{\varphi} \overline{\widehat{\bar{v}}} \in L_2(\mathbb{R}^d)$. This maps isometrically into $L_2(\mathbb{R}^d)$, because the canonical inner product of such functions is

$$(f_u, f_v)_\Phi = (2\pi)^{-d} \int \varphi \widehat{u} \overline{\widehat{v}}.$$

Now (6) carries over to the closure, i.e. the native space, and it yields the desired result. □

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